S.NO: 76 BATCH: 2014 – 2016 REG. NO.:

END OF SEMESTER EXAMINATIONS, NOVEMBER - 2018
TOPOLOGY AND FUNCTIONAL ANALYSIS
SUBJECT CODE: 14P3MA11

MAJOR: M.SC (MATHS) **SEMESTER : III**
TIME : 3 HOURS **MAX. MARKS: 70**

SECTION – A (5 X 4 = 20)

Answer All the questions:

1. Prove that a set U of a space X is open iff U contains a neighbourhood of each of its points.

(OR)

2. Prove that the projection $p_\mu : \phi \to X_\mu$ is an open map from ϕ onto X_μ for each $\mu \in M$.
3. Prove that the topological product X of any collection $\{X_\mu / \mu \in M\}$ of Hausdorff space is a Hausdorff space.

(OR)

4. Show that every continuous image of a compact space is compact.
5. Prove that the real line R is connected.

(OR)

6. If a space X is locally contractible at a point $p \in X$ then prove that X is locally path wise connected at the point p.
7. Define a normed linear space and Banach space with an example.

(OR)

8. State and prove the Open Mapping Theorem.
9. Let M be a closed linear subspace of a Hilbert space H, let x be a vector not in M, and let d be the distance from x to M. Then prove that there exists a unique vector y_0 in M such that $\| x - y_0 \| = d$.

(OR)

10. State and prove the Poparisation identity.

SECTION – B (5 X 10 = 50)

Answer All the questions:

11. For any given set E in a space X, prove that the following results:
    a) $cl(E) = E \cup \partial(E) = Int(E) \cup \partial(E)$
    b) E is open iff $Int(E) = E$
    c) E is closed iff $cl(E) = E$

(OR)

12. If $f : X \to Y$ is a function from a space X into a space Y with a given basis and a given sub-basis of its topology, then prove that the following statements are equivalent:
    (i) The function $f : X \to Y$ is a map.
    (ii) The inverse image $f^{-1}(U)$ of each open set U in Y is open in X.
    (iii) The inverse image $f^{-1}(V)$ of each basic open set V in Y is open in X.
    (iv) The inverse image $f^{-1}(W)$ of each sub-basic open set W in Y is open in X.
    (v) The inverse image $f^{-1}(F)$ of each closed set F in Y is closed in X.
    (vi) $f[cl(A)] \subset cl[f(A)]$ for each $A \subset X$.
    (vii) $f^{-1}[cl(B)] \supset cl[f^{-1}(B)]$ for each $B \subset Y$.

13. State and prove the Urysohn's lemma.

(OR)

14. State and prove the Tychonoff's theorem.

15. Prove that the topological product of an arbitrary family of connected spaces is connected.

(OR)

16. Prove that a space X is locally pathwise connected at a point p iff every neighbourhood of p contains a pathwise connected neighbourhood of p.

17. Let M be a closed linear subspace of a normed linear space N, If the norm of a coset $x+M$ in the quotient space N/M is defined by $\|x+M\| = \inf\{\|x+m\| : m \in M\}$ then prove that N/M is a normed linear space. Further, if N is a Banach space, then so is N/M.

(OR)

18. Let N and N' be normed linear spaces and T a linear transformation of N into N'. Then prove that the following conditions on T are all equivalent to one another:
   (1) T is continuous
   (2) T is continuous at the origin, in the sense that $x_n \to 0 \Rightarrow T(x_n) \to 0$.
   (3) there exists a real number $K \geq 0$ with the property that $\|T(x)\| \leq K\|x\|$ for every $x \in N$
   (4) if $S = \{x : \|x\| \leq 1\}$ is the closed unit sphere in N, then its image T(S) is a bounded set in N'.

19. Let $\{e_1, e_2, \ldots, e_n\}$ be a finite Orthonormal set in a Hilbert space H. If x is any vector in H, then prove that $\sum_{i=1}^{n} |(x_n e_i)|^2 \leq \|x\|^2$.

   Further, $x - \sum_{i=1}^{n} (x_1 e_i) e_i \perp e_j$ for each j.

(OR)

20. Let H be a Hilbert space, and let $\{e_i\}$ be an Orthonormal set in H. Then prove that the following conditions are all equivalent to one another:
   (1) $\{e_i\}$ is complete.
   (2) $x \perp \{e_i\} \Rightarrow x = 0$
   (3) if x is an arbitrary vector in H, then $x = \sum (x, e_i).e_i$
   (4) if x is an arbitrary vector in H, then $\|x\|^2 = \sum |(x, e_i)|^2$.

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